Closing horizontal groundwater fluxes with pipe network analysis: an application of the REW approach to an aquifer

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Abstract

The Representative Elementary Watershed (REW) approach for modelling the hydrologic response of watersheds is based on the discretization of a catchment into hydrologically sensitive control volumes. Global balance laws for mass, momentum (and energy) are formulated for the volumes. An implementation of the approach requires closing unknown REW-scale mass fluxes and forces exchanged across the REW-internal control volume boundaries and between REWs. In the present paper we focus exclusively on the horizontal groundwater flow, while neglecting other hydrological processes such as surface runoff, subsurface stormflow and surface-groundwater interaction. Here we describe a simplified groundwater modelling approach, which is based on the parsimonious estimation of mass fluxes exchanged laterally between REWs. The procedure employs principles of mass and energy conservation as stated by the Kirchhoff laws. The problem is reduced to the solution of a coupled system of linear equations in terms of inter-REW groundwater fluxes and a dimensionless ratio Θ between the local saturated zone depths $y^s$ and spatial length scales Λ over which piezometric head gaps are dissipated. The equation system is solved by successive approximation, as suggested by the Cross method, which is used in engineering practice for resolving resistor networks under steady state conditions. The procedure converges rapidly for complex network configurations. It is shown that for given boundary fluxes imposed at the watershed edges, mass fluxes exchanged in-between REWs can be calculated in an unique and consistent fashion. The paper briefly reviews the essential governing equations, introduces the formulation of the problem in terms of the Kirchoff laws and shows
the numerical solution of the network. Simulation results for an application to the Hesbaye groundwater system in Belgium are presented.
1 Introduction

The Representative Elementary Watershed (REW) approach introduced by Reggiani et al. [1998, 1999, 2000] and first implemented by Reggiani and Rientjes [2005] provides an alternative way of modeling the hydrological response of a river basin, including surface and subsurface runoff as well as surface-groundwater interaction. The principal novelty consists in partitioning a watershed into three-dimensional spatial entities (REWs), that are defined in an invariant fashion. REWs are recognizable at various spatial scales, ranging from the entire watershed to small areas of a few hectares. Within a REW a series of zones are defined on the basis of observable flow regimes. These zones are used as control volumes for the formulation of governing equations at the scale of the REW. The equations constitute Ordinary Differential Equations (ODEs), where the rate of change of a property, such as mass or momentum, are balanced by a series of fluxes or forces exchanged across the volume boundaries. Traditional physically-based distributed hydrological modeling approaches follow an approach, in which flow processes are described through partial differential equations (PDEs) to be solved within a 3-D model domain. The REW method, on the other hand, rests on global balance laws formulated directly at the spatial scale of the REW. Spatial gradients of various quantities are transformed via integration into mass or energy fluxes as well as forces acting on the boundaries of the control volumes. The principal advantage of the REW approach vs. models based entirely on the solution of PDE’s consists in substituting the simulation of spatial gradients of quantities such as mass or momentum over the domain by the determination of their respective fluxes across the domain boundaries. The REW
does not rely on fully distributed model domains with geo-located calculation nodes at the centroids of calculation units. This significantly reduces the computation effort due to use of ODE’s, making the approach also potentially suitable for large-scale hydrological applications requiring significant computational effort. Besides consistently addressing the issue of spatial scales of flow processes, the REW hydrological modeling framework has been developed with the idea to systematically address disparate temporal scales of flow processes through time integration. However, this latter aspect will not be used here, as horizontal groundwater flow will be studied as the only flow process. For additional detail the reader is referred to Reggiani et al. [1998].

Reggiani and Rientjes [2005] provide a hydrological application to an integrated hydrological system, whereby the groundwater modelling approach described here has been adopted and the description of the model has been kept at a minimum. The present paper is dedicated to describing the procedure and underlying assumptions and limitations in detail. Quite recently a preliminary application of the REW approach to integrated surface-groundwater systems has been published by Zhang and Savenije [2005] and Varado et al. [2006]. In both papers the solution of the groundwater flow equations is performed in the context of a REW hydrological modeling framework. However, the groundwater fluxes are closed on the basis of ad-hoc flux closure schemes, that do not invoke the conservation of mass and energy as it is proposed here via the Kirchoff laws for resistor networks.

The main challenge in implementing the REW approach consists in providing appro-
appropriate closure formulations for horizontal groundwater fluxes or forces via adequate parameterizations. In the hydrological literature (e.g. Beven [2006]) classical difficulties encountered when parameterizing the fluxes have been broadly addressed. Here we aim at providing a potential solution to the problem of process parameterization in closing the horizontal groundwater fluxes, that are exchanged laterally between REWs. The proposed closure approach is parsimonious and rests on first principles. A comprehensive procedure based on the Kirchhoff laws is introduced. Unknown groundwater fluxes as well as distances, over which piezometric heads are dissipated, must be estimated. The application of the procedure requires the assumption of slow flow which can be approximated as quasi-steady state. Under such assumption an aquifer separated into REW entities can be modeled in analogy to a resistor pipe network. The distribution of mass exchanges in the network can be resolved by applying the Cross [1936] load balancing method. The analogy of using pipe or resistor networks to describe groundwater flow is often found in the porous media literature. De Arcangelis et al. [1986] describe tracer motion in porous media by analogy to random tube networks. Other authors model multi-phase porous media flow through network models, as for example the one developed by Dias and Payatakes [1986]. In particular karstic systems are frequently conceptualized as interconnected pipe systems. An example is given in a paper by Liedl et al. [2003]. The principal underlying reason for introducing the proposed groundwater modelling approach is dictated by the need for a representation of surface and subsurface flows within an integrated hydrological modelling framework which is consistent with the level of spatial discretization of the REW.
Flow resistivity and piezometric heads at the network nodes are assumed as known, while the lateral mass exchanges and length scales, over which the piezometric head gaps are dissipated, constitute unknowns. The procedure is used under restrictive, however, plausible assumptions, that hold for the particular study case and for other, more general situations. Moreover, the procedure leads the way towards a systematic way of proceeding in providing closure for the REW-scale balance equations. The paper is organized in four main sections. In Section 2 the balance laws are revisited and the scientific problem is set into perspective. Section 3 addresses the closure of the forces and mass exchange terms, Section 4 the application of Kirchhoff laws for resistor networks, while Section 5 presents the actual application including some results for the Hesbaye aquifer in Belgium.

2 Geometry and flow equations

Reggiani and Rientjes [2005] provide a comprehensive description on how to model an integrated hydrological system with the REW approach, including surface flow as well as unsaturated and saturated groundwater flow. Here the hydrological system to be analyzed is limited to the sole saturated groundwater. The essential concepts are synthesized in two figures. Figure 1 shows a single REW including the global reference system and the horizontal area projection S.
**Figure 2** depicts a schematic transect of a REW, including the vertical mantle segments, through which groundwater in the saturated zone is exchanged with neighboring REWs. The mantle segments can also coincide with the external boundary of the watershed. The lower boundary of the REW is commonly formed by a limit depth and can in principle have any arbitrary shape. An average elevation $z^*$ of the lower boundary is indicated in **Figure 2** and is expressed with respect to the $z$-axis of the global reference system. The average water level within the REW is indicated with $y^*$. In **Figure 2** the real water table (dashed line) is drawn above/below the average water table (solid line). The average piezometric head in the saturated zone is expressed as follows:

$$h^* = y^* + z^*$$  \hspace{1cm} (1)

The position vector $\mathbf{x}^c$ (vectors are indicated in bold) of the REW centroid is introduced next. It is calculated through an area integral over the horizontal REW area projection $S$:

$$\mathbf{x}^c = \frac{1}{S} \int \mathbf{x} \, dS$$  \hspace{1cm} (2)

**Figure 3** shows the planimetry of a watershed separated into 5 REWs. The mantle segments forming the boundary between a REW and its neighbor, or those segments overlapping with the external boundary of the watershed, are clearly visible. The boundary segments separating two REWs are indicated with $C_{ij}$, while the segment forming the external boundary of REW $i$ is indicated with $C_{i \text{ ext}}$. The flow exchange area vector between two REWs across the mantle segment is given by the following
where $C_{ij}$ (bold) is the contour curve vector defined through the following integral along the contour curve $C_{ij}$:

$$C_{ij} = \int_{C_{ij}} n \ dC$$

with $n$ the vector normal to the curve pointing outward. The quantity $y_{C_{ij}}$ is the average thickness of the saturated zone evaluated over the mantle segment. The symbol $C_{ij}$ is naturally also used to denote the absolute length of the contour vector. These variables and respective geometrical concepts are essential for the introduction of the proposed procedure. They will be used throughout the following parts of the paper. In the following paragraphs the mass and momentum conservation laws are presented. These are significantly simplified with respect to the full development. For a more complete overview the reader is referred to Reggiani and Rientjes [2005].

**Mass conservation**: The saturated zone mass conservation for a REW is stated as follows:

$$S \ \epsilon \ \rho \ \frac{dy^s}{dt} = \Sigma_j \ \epsilon_{ij}^{sm} + \epsilon_{su} + \epsilon_{i \ ext}^{sm}$$

where $S$ is the horizontal surface area of the REW, $\epsilon$ is the porosity, $\rho$ is the mass density of water and $y^s$ is the average vertical thickness of the saturated zone. The
right hand side terms are the sum of mass fluxes across the individual mantle seg-
ments, $e_{ij}^{sm}$, and the recharge or capillary rise flux across the water table, $e^{su}$. The
flux across the external watershed boundary (if non-zero) is denoted with $e_{iext}^{sm}$.

**Momentum conservation**: The balance equation for momentum in the saturated
zone, under steady state assumptions, is given by the following vectorial equation:

$$-S \gamma^s \epsilon \rho \mathbf{g} = \Sigma_j T_{ij}^{sm} + T^{sbot} + T^{su} + T_{iext}^{sm} - R \mathbf{v}^s$$  \hspace{1cm} (6)

where, $\mathbf{g}$ is the (vectorial) gravitational acceleration and $\mathbf{v}$ is the Darcy velocity.
The terms on the r.h.s. are the acting forces and represent, in order of appearance,
the sum of forces acting on each individual mantle segment, $T_{ij}^{sm}$, the force acting
on the lower boundary of the saturated zone, $T^{sbot}$, and the force exchanged with
the unsaturated zone across the water table, $T^{su}$. $T_{iext}^{sm}$ is the force acting on the
external watershed boundary (if non-zero) for that REW. $R$ is the flow resistivity
of the aquifer material:

$$R = \frac{\gamma^s g \rho S}{K}$$  \hspace{1cm} (7)

The momentum conservation equation is vectorial and must be projected along
the axes of the reference system. This is achieved by scalar multiplication of the
equations with the unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ pointing along the axes of the reference
system in **Figure 1**. The vertical component of Eq. (6) reduces to a static balance
of forces, while the horizontal components yield the following algebraic equation:
with $\lambda = x, y$. For the evaluation of the inter-REW groundwater fluxes, only the mass balance equation (5) is actually needed, while the momentum balance (8) is reported here for completeness. It will be used for the solution of the horizontal velocity field in the aquifer.

### 3 Closure

The mass fluxes and forces appearing in Eqns. (5) and (8) need to be closed by means of appropriate parameterizations. The way has been led in previous papers by the first author where preliminary working parameterizations have been provided. With respect to the mass exchange terms, these are usually linearized in terms of the total piezometric head differences between zones internal to REW and REWs. The forces are expressed as an average (hydrostatic) pressure, integrated over the respective exchange surface, such as a mantle segment, whereby the respective surface areas must be approximated. In the following paragraphs we show how the unknown terms in the equation of mass conservation (5) and the momentum equation (8) can be closed.

#### 3.1 Groundwater recharge

The net flux across the water table $e^{su}[MT^{-1}]$ in (5) is attributable to percolation and/or capillary rise, and is controlled by the vertical water movement in the
unsaturated zone. The flux switches from positive during recharge of the aquifer, to negative when mass is diffused upwards through capillary rise. The quantity $e^{su}$ has been simulated in a preliminary fashion in Reggiani et al. [2000]. Here $e^{su}$ is estimated through solution of Richards equation (Ross [2003]), with a moving free-drainage boundary at the phreatic surface separating the unsaturated and saturated zones. A mean annual flux $E[e^{su}]$ is introduced, which represents the long-term average of $e^{su}$. Both quantities represent local recharge rates integrated over the phreatic surface in a REW.

### 3.2 Lateral mass fluxes

The mass exchange between a REW $i$ and its $j$-th neighbor is driven by the difference in piezometric heads across the mantle segment:

$$
e^{sm}_{ij} = \alpha_{ij} (h^s_j - h^s_i)$$  \hspace{1cm} (9)

where the linearization coefficient $\alpha_{ij}$ is defined as follows:

$$\alpha_{ij} = \bar{K}_{ij} C_{ij} \frac{y^{C_{ij}}}{\Lambda_{ij}}$$  \hspace{1cm} (10)

with $\bar{K}_{ij}$ an average hydraulic conductivity between two adjacent REWs. The quantity $\Lambda_{ij}$ is an unknown length scale over which the piezometric head difference is dissipated. This length scale is usually unknown and cannot be assumed to be a geometrical surrogate quantity such as for example the straight-line distance between two REW centroids. The straight line distance could for instance be equal to zero for cases in which the centroids of two REWs coincide, as infinite REW
shapes are in principle possible. Zero-length dissipation lengths are however non
physical and unjustifiable in practice. To avoid a dissipation of head differences
over an (artificially) too short inter-REW distance for such particular situations, $\Lambda_{ij}$
needs to be determined on the basis of mass and energy conservation. In this fashion
the procedure of flux estimation becomes consistent, objective and parsimonious.
To facilitate the presentations of the envisaged method the various quantities in
(10) are grouped into two parameters:

$$\beta_{ij} = K_{ij} C_{ij}$$  \hspace{1cm} (11)

lumps together known geometric properties, such as the contour length and the
hydraulic conductivities, while

$$\Theta_{ij} = \frac{y_{C_{ij}}}{\Lambda_{ij}}$$  \hspace{1cm} (12)

represents an unknown quantity, namely the ratio of the water table elevation at
the mantle segment centroid over the dissipation length scale. The inter-REW mass
flux (10) can be rewritten more concisely as follows:

$$\epsilon_{ij} = \beta_{ij} \Theta_{ij} (h_j - h_i)$$  \hspace{1cm} (13)

For a piezometric head difference $h_j > h_i$, mass is added to REW $i$ yielding a
positive mass exchange term. Otherwise REW $i$ discharges towards the neighbor $j$
resulting in a mass loss for that REW.
3.3 Horizontal forces

The horizontal forces in Eq. (8) used for the calculation of the flow velocity $v^s$ are expressed as the product of the hydrostatic pressure acting on the mantle segment with its surface area vector. The respective components of the forces along the horizontal axes of the coordinate system are obtained by taking the inner product with the unit vector $e_\lambda$:

$$\langle T_{ij}^m + T_{i ext}^m \rangle \cdot e_\lambda = \frac{\rho g}{2} (y^{C_{ij}} A_{ij} + y^{C_i ext} A_{i ext}) \cdot e_\lambda$$  \hspace{1cm} (14)

4 The Kirchhoff Laws

For a given recharge rate $e_{su}$ the simultaneous solution of the mass balance equation (5) for the ensemble of REWs gives the average water table position $y^s$ for each individual REW, whereas (8) provides the horizontal aquifer velocity field. Once the hydraulic conductivities and REW geometry represented by $\beta_{ij}$ are known, the length scale $\Lambda_{ij}$ remains the only unknown quantity. It is represented by the ratio $\Theta_{ij}$, that can be estimated parsimoniously through the procedure shown next.

For the calculation of $\Theta_{ij}$ an equation system based on the Kirchhoff laws for resistor networks is proposed. One can in principle assume that the aquifer is at any time sufficiently close to the steady state (at a quasi-steady state). This assumption is reasonable for REWs which have a shallow groundwater table and are fast-reacting, thus evolving rapidly (e.g. within the duration of a model time step) towards a new equilibrium in a succession of steady states. As in Figure 4, the aquifer separated
into a finite number of REWs can be regarded as a network of pipes of length $\Lambda_{ij}$ linking a series of nodes.

**INSERT FIGURE 4 HERE**

The position of the nodes coincides with a "centre" location within (or in some cases also outside) the REW, which is not fixed, but may change between calculation steps. The relative distance of these points is expressed via the pipe length or distance $\Lambda_{ij}$. The pipes represent REW interconnections and identify the number $NL$ of non-redundant loops via the following recursive formula:

$$NL = NP - (NN - 1)$$

(15)

with $NP$ the number of pipes and $NN$ the number of network nodes. With reference to Figure 4 the first Kirchoff law for a network node can be stated, which requires that the sum of mass fluxes entering (positive) and exiting (negative) at the node must add up to zero (mass conservation). In the case of REW 1 the total mass exchanged with REWs 2 and 4 minus external demand plus external supply must balance each other:

$$e_{sm}^{11} - e_{sm}^{12} + e_{su}^{1} - e_{sm}^{1 ext} = 0$$

(16)

The second Kirchoff law states that the sum of head losses along a closed loop must equal zero (energy conservation). For example in loop $1 - 2 - 4$ we state the sum of total piezometric head losses as follows:
\[
\Delta h_{11}^* + \Delta h_{12}^* + \Delta h_{24}^* = 0
\]  
(17)

where \( \Delta h_{ij}^* = (h_i^* - h_j^*) \). The heads constitute known quantities of the problem and are calculated through the mass balance (5).

By applying the Kirchoff laws to the pipe network for given supply and demand at the nodes, a discharge distribution can be found, which is in equilibrium with the current distribution of piezometric heads at the nodes and the supply/demand rates. This procedure of finding equilibrium discharges is routinely used in engineering practice and proposed by Cross [1936] for the design of urban water distribution networks. To illustrate its application for the estimation for inter-REW groundwater fluxes, we apply the procedure to the network in Figure 4. The conservation of mass at the five nodes is stated in matrix form:

\[
\begin{bmatrix}
-1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\epsilon_{12}^{sm} \\
\epsilon_{14}^{sm} \\
\epsilon_{23}^{sm} \\
\epsilon_{24}^{sm} \\
\epsilon_{34}^{sm} \\
\epsilon_{35}^{sm} \\
\epsilon_{45}^{sm} \\
\end{bmatrix}
= 
\begin{bmatrix}
\epsilon_{1}^{su} - \epsilon_{1}^{sm} \\
\epsilon_{2}^{su} \\
\epsilon_{3}^{su} \\
\epsilon_{4}^{su} - \epsilon_{4}^{sm} \\
\epsilon_{5}^{su} - \epsilon_{5}^{sm} \\
\end{bmatrix}
\]  
(18)

These equations constitute a linear system with \( NN-1 \) independent equations. The conservation of energy for each loop yields additional NL (i.e. three) equations:
The two systems (18) and (19) have thus a total number of $NP = NL + NN - 1$ independent equations with $2NP$ unknowns, namely the mass exchange terms and the ratios $\Theta_{ij}$. For assumed initial values of $\Theta_{ij}$ a unique and non-degenerate solution can be found by applying an iterative procedure. First, an initial vector of unknown mass exchange terms is assigned, that satisfies the continuity at the nodes (18), but not necessarily the conservation of head losses (19). In addition, guessed initial values (e.g. equal to 1) for $\Theta_{ij}$ are chosen at the start of every iteration cycle. The guessed initial discharge distribution for a loop is improved iteratively by adding a correction at each step. Adding the increment clockwise to all pipes in the loop assures mass conservation at the nodes, thus satisfaction of (18) at each iteration step. Explicitly, for loop $1-2-4$ in the first row of (18), the mass exchange correction at iteration $n + 1$ is calculated via the following expression:

$$\pm|\Delta|^{n+1} = \frac{e_{12}^{sm}}{\Theta_{12}\beta_{12}} + \frac{e_{14}^{sm}}{\Theta_{14}\beta_{14}} + \frac{e_{24}^{sm}}{\Theta_{24}\beta_{24}}$$

(20)
which after addition to the mass exchange terms yields corrected inter-REW mass
exchange distribution at iteration $n + 1$. For the updated distribution new coeffi-
cients $\Theta_{ij}^{n+1}$ are obtained via (13). The iterative procedure of alternatively calcu-
lating mass exchange corrections and coefficients is repeated, until a convergency
criterion on the mass exchange corrections is satisfied:

$$\left[ \sum_{NP} (|\Delta|^{n+1})^2 \right]^{\frac{1}{2}} < \text{tolerance} \quad (21)$$

A final distribution for $\Theta_{ij}$ and $e_{ij}^{sm}$ is obtained. The unknown length scales $\Lambda_{ij}$ can
be calculated by performing a 2-D cubic-spline interpolation (Inoue [1986]) from the
REW-averaged water table elevations towards the centroids of the contour curves
$C_{ij}$, yielding suitable approximations for $y^{C_{ij}}$ and subsequently the $\Lambda_{ij}$ values via
(12). The interpolated water table and the approximated saturated zone depths
$y^{C_{ij}}$ are both shown in Figure 2.

5 Application

5.1 Study basin analysis

The outlined procedure is applied and tested on the 494 km$^2$ basin of the river Geer
in Belgium, a tributary of the river Meuse. The basin planimetry and its separation
into REWs is depicted in Figure 5. The basin is underlain by a deep ground-
water system in porous chalk layers, that is delimited at the bottom by a slightly
northwards inclined near-impermeable smectite substratum (Dassargues and Mon-
joie [1993]).
The selected study site is well suited and satisfies the assumptions of slow flow and other important criteria for the implementation of the proposed flux balancing method, such as clearly defined REW mantle surfaces and known external boundary fluxes. The extraction of REWs is performed with a special module, which has been added to the digital elevation map analysis suite Tardem (Tarboton et al. [1997]). The module calculates the detailed geometry of the REWs, including planar and curved surface areas and the contour segments $C_{ij}$ in Figure 3 which are required for the approximation of the mantle areas. The projection of the respective contour segments onto the reference system axes required in Eq. (14) are also calculated and the connectivity between REWs is established. A Strahler network order equal to 2 has been chosen as threshold value for the definition of the REWs, yielding a total number of 73 modeling entities. For further details on the procedure, the reader is referred to Tarboton et al. [1997]. For the setup of the network, an appropriate module for pipe mesh analysis has been developed. The module assembles a consistent network of nodes and pipes by sweeping all REWs, verifying lateral inter-REW connectivities and identifying closed loops. At a node as many pipes converge, as there are neighboring, interconnected REWs (i.e. mantle segments with non-zero surface area). Numerical tests with various watersheds at various levels of discretization showed that two types of loop structures are possible: $i)$ triangular loops with three nodes and three pipes and $ii)$ quadrangular loops with four nodes and as many pipes. Once the loops are formed, redundant loops, that have more
than two pipes or more than three nodes in common, need to be eliminated. Finally, all loops are oriented clockwise to assure correct algebra when balancing discharges iteratively. The basin separated into 73 REWs with a network of $NN = 73$ nodes and $NP = 178$ pipes yields a total number of $NL = 106$ non-redundant loops. The resulting linear system has 178 equations in as many unknown inter-REW mass exchanges. A list of typical network characteristics for three possible spatial discretization levels based on the Strahler ordering system are summarized in Table 1.

<table>
<thead>
<tr>
<th>Strahler order</th>
<th>No. of REWs</th>
<th>No. of Pipes</th>
<th>No. of Loops</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>463</td>
<td>1176</td>
<td>741</td>
</tr>
<tr>
<td>2</td>
<td>73</td>
<td>178</td>
<td>106</td>
</tr>
<tr>
<td>3</td>
<td>17</td>
<td>32</td>
<td>16</td>
</tr>
</tbody>
</table>

Table 1: Network characteristics for the Geer basin

5.2 Supply and demand

The Cross network balancing method holds for a steady state regime. The sum of all discharges exiting a node must balance those entering. Thus no mass is lost or added to the system during the iterative procedure of mass-exchange balancing in the network loops. It is important to note that in practice the state of the hydrological system can very well depart from steady state conditions. It is forced dynamically by mass input (precipitation) during storms or mass extraction (evapotranspiration) during inter-storm periods. Internally, water is continuously re-distributed among
REW$s$ through mass exchange determined by velocity fields and piezometric head gradients. Moreover, a system can have permeable boundaries and thus loose or receive water from outside the groundwater basin boundaries, as is the case for the Hesbaye aquifer.

Given that matrix flow conditions prevail and the flow in the aquifer is in the order of 0.1 m/day (Dassargues et al. 1988), it is assumed that the inter-REW mantle fluxes (13) are driven by piezometric head differences, which are updated dynamically via the mass balance (5) whereby the ratios $\Theta_{ij}$ in (13) are kept constant. The ratios $\Theta_{ij}$ are updated only periodically under steady state conditions, while in reality they may change continuously in time. The ratios are updated during the simulation by "freezing" the piezometric heads at regular intervals, re-balancing the mass exchanges in steady state, and then resuming the solution of (5) under dynamic conditions with new $\Theta_{ij}$ values. Updating the $\Theta_{ij}$ ratios can be effectuated frequently, but from experience gained for this application, the values vary moderately between updates over the domain. This suggests that a low updating frequency with respect to model time step is appropriate.

Next, demand and supply $e_{i\text{ext}}^{sm}$ and $e^{su}$ in the linear system (18) are imposed at the network nodes. For the present application it has been decided to set the supply rates $e^{su}$ at the network nodes equal to the mean annual recharge rate $E[e^{su}]$. The mean annual recharge is estimated from geological surveys which have established the water balance of the Hesbaye aquifer over the observation period between 1975
and 1999 (Boygues et al. [2004]) as follows:

\[
E[P] = E[ETR] + E[Q_{riv}] + E[Q_{Out}] + E[Q_{bound}] 
\] (22)

whereby the various quantities, expressed in millimeters per year, are:

\[
810 \text{ mm/yr} = 508 + 145 + 69 + 88 \text{ mm/yr} 
\] (23)

The l.h.s. term is the mean annual precipitation, whereas the first two terms on
the r.h.s. are the average annual evapotranspiration and the catchment runoff. The
last two terms constitute annual losses for the groundwater system: the extraction
through pumping wells and drainage galleries, \(Q_{out}\), and the loss across the catch-
ment boundary \(Q_{bound}\). The leakage is occurring mainly along the northern bound-
ary, as confirmed by geological surveys. With reference to Figure 5 groundwater ab-
straction on the catchment boundary is allowed for REWs 9, 12, 13, 25, 26, 27, 39, 52, 68, 69
and 73. The mean annual loss is extracted as demand on the external boundaries
of those REWs by weighting the total mean annual loss by the respective external
mantle areas. In applying the mean annual loss, the pumping rates and gallery
abstractions have been implicitly added to the boundary losses, thus preserving the
mean annual water balance (22). This is done here for mere reasons of simplic-
ity, without applying the pumping rates in a spatial distributed manner over the
catchment.

Finally, the mean annual recharge \(E[e^{su}]\) is obtained from (23) by multiplying the
mean net recharge with the area \(S_i\) of REW \(i\)
\[ E[e^{su}] = E[P - ETR - Q_{riv}] S_i \]  

(24)

In this manner one obtains supply rates for each REW which preserve the mean annual water balance and thus the long-term steady state situation within the network. Once a new distribution of \( \Theta_{ij} \) is found via iteration, the mass exchange terms (13) can be updated, and the solution of the mass balance (5) continued under dynamic conditions, with instantaneous recharge fluxes \( e^{su} \) calculated via the solution of Richards equation.

### 5.3 Results and Discussion

The methodology is implemented as outlined above, and subsequently tested by forcing the basin over the 10-year period 1/1/1985 - 31/12/1994. Daily rainfall observation series from the gauging station network in Figure 5 are mapped through kriging towards the REW centroids. The total mean annual supply rate \( E[e^{su}] \) at the network nodes is estimated via (24) and abstracted from the REWs with a permeable boundary on the northern watershed boundary. The REWs sharing the remaining part of the external basin boundary are considered impermeable. The solution of the mass balance equation (5) is performed with an adaptive step-size Runge-Kutta solver (Press et al. [1994]). The maximum calculation step-size is one hour. The ratios \( \Theta_{ij} \) are updated every 7 days during the simulation period, by "freezing" the piezometric pressure heads and subsequently re-balancing the inter-REW mass exchanges over the network loops. The 7 days period has been selected, since we consider a week to be sufficient for reaching successive equilibria under...
quasi-steady state conditions (see also Section 4). For an elimination of initial-
condition effects the model is run first during a ”warm-up” period of 10 consecutive
years prior to the start of the simulation period. We assume this time span to be
sufficiently long to reach an equilibrium between the external forcing and the water
redistribution within the system. The values of the saturated zone depth at the
end of the warm-up run are used as new initial conditions for the actual simulation
period. The most relevant results are reported in Figures 6-9.

For reasons of presentation, a transect through the catchment has been selected.
The transect is represented by the dashed line in Figure 5 and samples REWs that
lie between the higher southern boundary and the lower areas closer to the river.
Selected dynamic variables and fluxes for REWs 13, 15, 16, 17 and 18 on the transect
are presented. Figure 6 shows the net recharge flux across the water table for the
simulation period. In the upper part of the figure the net atmospheric forcing on
the surface, i.e. the recorded rainfall \( P [mm/day] \) minus the potential evaporation
\( PE [mm/day] \) (Penman-Monteith method) at the gauging station Bierset, is plot-
ted. The non-dimensional net water table recharge for REWs 13, 15, 16, 17 and 18
is shown in the lower half of the figure.

The actual water removal at the soil surface is limited by the diffusive capacity of
the unsaturated soil column to deliver moisture to the surface (stage-2 evaporation,
Ritchie [1972]) and is thus significantly inferior to the locally applied potential rate
(stage-1 evaporation). The hydraulic conductivity of the soil is sufficiently large

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\( K = 5 \times 10^{-4} \text{m/s} \) to prevent the formation of infiltration excess runoff on the surface. The moisture extracted from the water table by capillary rise does not reach the surface because of the substantial depth to the water table which can reach up to 40 m in some parts of the basin; an exception are the limited areas in the vicinity of the stream network. As to be expected the recharge flux signal \( e_{su} \) is smoothed with respect to the forcing signal on the land surface. This effect is attributable to the redistribution of moisture across the unsaturated zone soil column. The modeled re-charge flux remains always positive, as the Richards equation solver has been implemented with a lower boundary condition based on free drainage. This choice has been motivated by the larger calculation time-steps that become possible in absence of a lower boundary condition which does not allow the water table flux to switch between re-charge and capillary rise. The recharge flux \( e_{su} \) computed with Richards equation is then used as instantaneous input for the solution of the mass balance (5), while the network topology (\( \Lambda_{ij} \) via \( \Theta_{ij} \) and \( y_{C_{ij}} \)) is updated periodically under steady state conditions with the estimated long-term mean recharge \( E[e_{su}] \) as supply.

**Figure 7** shows the non-dimensional total net mass flux \( \sum_i e_{sm_i} \) in Eq. (5) across the REW mantle. The fluxes crossing the individual mantle segments, including - if applicable - the fluxes across the external watershed boundaries, are added up to a total net mantle flux. During the simulation period the net flux can switch sign, implying that the REW water balance changes from net recharge through lateral recharge from its neighbors, to net discharge through lateral transfer towards its
neighbors. From the figure we see that REWs 14, 15 and 16 have a predominantly positive total mantle flux indicating that they receive water laterally from neighboring REWs, while REWs 13, 17 and 18 are characterized by a negative mantle flux, indicating that they are loosing water towards their neighbors. REW 13 is a boundary REW, where groundwater is allowed to leave the system, and thus a negative mantle flux is to be expected. Overall groundwater is moving from the higher-lying southern boundary (REW18) towards the lower-lying permeable northern boundary, in agreement to actual observations and simulations by Dassargues et al. [1988].

It is recalled that the lateral mantle fluxes $e_{ij}^{sm}$ are calculated via expression (13), which involves re-evaluating the network topology in terms of the dissipation length $\Lambda_{ij}$ at each discharge re-balancing step.

**Figure 8** compares the modeled water table responses for the selected REWs with measured piezometric levels for the four piezometric wells F06, F29, ORTH022 and WAL66 indicated in **Figure 5**. The plots have been restricted to the years for which observations were available. Generally the water table fluctuations measured at the wells present a good match with the average fluctuations simulated for REWs 39 (F06), 48 (F029), 49 (WAL66), and 51 (ORTH022), respectively. We note that conceptually the REW approach calculates average water levels for an entire REW, that can only be compared in particular situations with the measured piezometric levels (that represent indeed point values). Moreover the Hesbaye aquifer is exploited by pumping and two drainage galleries in the aquifer, which influence the shape of the water table locally. Reproducing its features accurately requires the
use of a high-resolution 3D aquifer model as the one described by Dassargues et al. [1988]. It is emphasized that it is not aim of the REW aquifer modeling approach to accurately represent localized phenomena, but rather to reproduce hydrological phenomena at the REW-average scale. Localized water table behavior cannot be captured by the approach.

Figure 9 shows the water table interpolated via the Inoue [1986] optimal surface fitting method. The water table surface is obtained through a bi-cubic spline interpolation across the REW-average piezometric heads, evaluated through the mass balance equation (5) and conceptually located in correspondence of the REW centroids. The method fits a surface through the "observation" points by minimizing the "elastic" energy of the surface discretized by means of quadrangular finite elements. The method is sensitive to the choice of a surface roughness parameter and a tension parameter which ultimately determines the smoothness of the fit and thus the shape of the interpolated surface. For the current simulations an Inoue (1986) surface roughness of 0.01 and a tension parameter of 0.95 have been selected. The piezometric surface in Figure 9 was obtained from a 100 x 100 element mesh. The contour curve gradients reflect the shape of the piezometric surface computed by Dassargues et al. [1988] with a 2670 8-node iso-parametric brick finite element model. However, given the coarse discretization of the aquifer in terms of few elements (73 REWs) the reproduction of the piezometric surface cannot be captured with the same accuracy to be achieved with more sophisticated numerical methods at high spatial resolution. The velocity field for each REW is calculated from the
solution of the algebraic equation (6) by making use of the actual water table elevations in correspondence of the mantle segments, including the external boundaries. An image of the computed groundwater flow field vectors for the Hesbaye aquifer has been reproduced in Reggiani and Rientjes [2005].

6 Conclusions

This paper introduces a parsimonious procedure for the estimation of inter-REW groundwater fluxes, by applying the Cross method used for the solution of resistor or water distribution networks. The outlined procedure is commonly used for calculating discharge distributions in pipe networks, whereby the geometric and hydraulic properties of the conducts, the demands and supply rates, as well as the network topology, are known. In such a procedure usually the distribution of piezometric heads at the network nodes represent the unknown quantities, which need to be determined. However, in the present application the piezometric heads are assumed to be known at a given point in time. These are calculated from the solution of the REW-average mass balance equations. The geometry of REW mantle segments and the hydraulic conductivities are also known and estimated from topographic analysis and field data. The unknown quantities remain the length scales $\Lambda_{ij}$, representing the network topology, over which the inter-REW piezometric head differences are dissipated, and the mass exchanges $e_{ij}^{sm}$ representing the inter-REW groundwater fluxes. The outlined procedure allows to estimate the fluxes unambiguously, by assuming steady state conditions at a given point in time and recalculating the co-
efficients $\Theta_{ij}$. It is important to note that the calculation of the length scale as an unknown quantity allows for the possibility of mass transfer between two REWs to occur either over a short direct distance or over longer tortuous paths. In this way the procedure accommodates for a range of possible unknown flow paths lengths, which depend on the particular configuration of boundary conditions, piezometric heads and hydraulic conductivities. At every updating, the path lengths are recalculated based on first principles and in a conservative fashion. We note that an explicit estimation of the dissipation lengths as performed here is not possible with the ad-hoc closure approaches for horizontal groundwater fluxes adopted by Zhang and Savenije [2005] or Varado et al. [2006]. The procedure shows through application to a natural aquifer system how REW-scale hydrological fluxes can be estimated consistently for practical applications in simulating hydrologic responses of catchments. The resulting simulations preserve the water balance of the system and reproduce the trends of groundwater movements measured at selected piezometric wells, including the overall shape of the head surface. The presented model calculates REW-average head surface values, that may differ from local piezometric head observations. The proposed approach is not aimed as an alternative to more detailed aquifer models such as MODFLOW or FEFLOW which are necessary for accurate representation of the phreatic surface at scales smaller than the sub-catchment scales employed by the REW approach. Instead it constitutes a sound proposition for modeling large groundwater systems in combination with a detailed process-based hydrological model for unsaturated zone and surface flow processes for hydrologic response simulations of watersheds. The simplified approach requires
limited computational effort and therefore allows for large-scale hydrological applications in combination with computationally demanding applications such as global circulation models and landsurface-atmosphere interaction schemes.

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8 References


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9 Table and Figure Captions

Table 1: Network characteristics for the Geer basin.

Figure 1: 3-D view of a REW, showing the horizontal area projection and the global reference system.

Figure 2: Transect through the REW, showing the principal geometrical definitions.

Figure 3: Birds eye view on a watershed separated into 5 REWs, including relevant geometrical information.
Figure 4: The pipe network for a watershed separated into 5 REWs.

Figure 5: The Geer Basin in Belgium (494 km2) separated into 73 REWs. The blue squares indicate the position of the rainfall gauging stations. The transect is visible in the figure.

Figure 6: Non-dimensional water table recharge fluxes and net precipitation for selected REWs and the simulation period 1985-1994.

Figure 7: Net horizontal inter-REW fluxes for selected REWs and the simulation period 1985-1994.

Figure 8: Water table positions compared with measured piezometric heads for selected REWs and the simulation period 1985-1994. The crosses represent the observed heads.

Figure 9: Piezometric head contours for the interpolated water table for day 31/01/1987.
Figure 2

Datum elevation

Average bottom boundary

Saturated zone

Recharge flux

Interpolated water table

Real water table

Mantle surface

Mantle flux
Figure 3
Figure 6

Non-dimensional vertical WT recharge flux $\frac{\text{e}^{\text{in}}}{\text{max}|\text{e}^{\text{in}}|}$ (-)

REW 13, average WT pos 100.3 m
REW 14, average WT pos 101.2 m
REW 15, average WT pos 119.3 m
REW 16, average WT pos 142.8 m
REW 17, average WT pos 154.4 m
REW 18, average WT pos 150.6 m
Non-dimensional horizontal inter-REW mantle fluxes, years 1985-1994

Figure 7

mass flux \( e^{in} / \max|e^{in}| \) (-)
Figure 8